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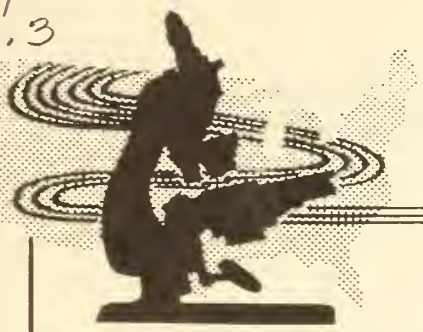

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Power Series Solutions of the One-Dimensional FLOW EQUATION for Exponential and Linear Diffusivity Functions

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POWER SERIES SOLUTIONS OF THE ONE-DIMENSIONAL FLOW EQUATION FOR EXPONENTIAL AND LINEAR DIFFUSIVITY FUNCTIONS[1/]

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INTRODUCTION

A number of transport phenomena are reasonably well described by a combination of a flux equation and the equation of continuity. Among these are the diffusion of matter, the flow of heat, and the flow of water in porous media.

In the flow of water in porous media, the flux equation that applies is Darcy's law (2).[3/] The combination of Darcy's law and the continuity equation leads to the equation:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K(\theta) \quad \nabla H) \quad . \qquad [1]$$

Where θ is the volume water content, H is the hydraulic head, K (θ) is the conductivity as a function of water content, and t is time (2). There are a number of assumptions inherent in the use of equation 1. It is not the purpose of this report to examine these. Items (2 and 3) in the Literature Cited contain additional material on this aspect of soil water flow.

Equation 1 can be modified as follows:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta) \quad \nabla \theta) + \frac{\partial K(\theta)}{\partial z} \qquad [2]$$

where the soil water diffusivity, D (θ), is given by:

$$D(\theta) = \frac{K(\theta)}{\frac{d\theta}{dh}} \quad . \qquad [3]$$

The pressure head, h, has been assumed to be a single valued function of θ.

1/ Contribution from the Soil and Water Conservation Research Division, Agricultural Research Service, USDA, and the Illinois Agricultural Experiment Station.

2/ Assoc. Prof. of Mathematics, University of Illinois; Investigations Leader, USDA, Fort Collins, Colo.; Soil Scientist, USDA, Urbana, Ill., and Assoc. Prof. of Soil Physics, University of Illinois; and Prof. of Soil Physics, University of Illinois; respectively.

3/ Italic numbers in parentheses refer to Literature Cited at the end of this publication.

If within the flow system, consumption or production of water occurs, this may be treated by adding another term in equations 1 or 2 so that they become:

$$\frac{\partial \theta}{\partial t} = \nabla \cdot (K(\theta) \nabla H) + A(x, y, z, t) \qquad [4]$$

$$= \nabla \cdot (D(\theta) \nabla \theta) + \frac{\partial K(\theta)}{\partial z} + A(x, y, z, t) \qquad [5]$$

where the function A (x, y, z, t) will be denoted as the "source term".

It is the purpose of this report to describe a power series method of solution of a one-dimensional form of equation 2 where:

(1) The flow is in the horizontal direction and

(2) The diffusivity is either an exponential or a linear function of water content.

Then equation 2 becomes:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta) \frac{\partial \theta}{\partial x}\right) \quad . \qquad [6]$$

The Exponential Diffusivity Function.

Experimental evidence gives some indication (1) that the diffusivity function for many soils can be characterized by an exponential function of the form

$$D(\theta) = \alpha e^{\beta \theta}, \ \alpha > 0, \ \beta > 0 \qquad [7]$$

where α and β are parameters. Using this function, we shall solve the following boundary value problems:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\alpha e^{\beta \theta} \frac{\partial \theta}{\partial x}\right), \quad 0 < x < \infty, \ t < 0, \qquad [8]$$

$$\theta(0, t) = \theta_k \quad (k = 1, 2), \ t > 0 \qquad [9]$$

$$\theta(x, 0) = \theta_o \quad x > 0,$$

$$0 < \theta_2 < \theta_o < \theta_1 \quad . \qquad [10]$$

The problems corresponding to k = 1 and k = 2 will be termed the Horizontal Infiltration Problem and the Horizontal Outflow Problem, respectively.

Despite the similarity of the two problems, the corresponding solutions exhibit distinctly different characteristics.

By introducing the dimensionless parameters $\zeta = x/L$, $\tau = \alpha t/L^2$ and making the substitution:

$$\theta = \frac{1}{\beta} \ln U, \qquad [11]$$

equation 8 becomes:

$$\frac{\partial U}{\partial \tau}(\zeta, \tau) = U(\zeta, \tau) \frac{\partial^2 U}{\partial \zeta^2}(\zeta, \tau) \quad . \qquad [12]$$

L is an arbitrarily chosen characteristic length different from zero. Solutions of equation 12 of the form $U(\zeta, \tau) = f(\eta)$ will be sought, where $\eta = \zeta / \sqrt{2\tau}$. Then equation 12 reduces to an ordinary nonlinear differential equation and the preceding problems become:

$$f(\eta)\, f''(\eta) = -\eta\, f'(\eta) \qquad [13]$$

$$f(0) = e^{\beta \theta_k}, \quad (k = 1, 2) \qquad [14]$$

$$f(\eta) = e^{\beta \theta_o} \qquad [15]$$

$$0 < \theta_2 < \theta_o < \theta_1 \quad . \qquad [16]$$

The resulting two-point boundary value problems pose difficulties because the condition of equation 15 is not easy to satisfy. Fortunately, the differential equation 13 and the initial conditions are such that each of the family of curves satisfying the latter tends to a horizontal asymptote as $\eta \longrightarrow \infty$. Hence, the problem can be transformed to an initial value problem by replacing the condition, as shown in equation 15, by:

$$f'(0) = \omega \neq 0 \quad . \qquad [17]$$

By varying the parameter ω, the condition of equation 15 can be satisfied for various values of θ_o, provided the initial value problem can be solved. This solution is possible by employing the Taylor series method. To that end, the nth derivative of equation 13 at $\eta = \eta_o$ was evaluated, and the resulting equation was solved for the derivative of order (n + 2):

$$f^{(n+2)}(\eta_o) = -\left[\frac{\eta_o f^{(n+1)}(\eta_o) + n f^{(n)}(\eta_o) + \sum_{k=1}^{n} \binom{n}{k} f^{(k)}(\eta_o)\, f^{(n-k+2)}(\eta_o)}{f(\eta_o)}\right] \qquad [18]$$

$$n = 1, 2, 3, \text{----}$$

The Taylor series solution is of the form:

$$f(\eta) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\eta_o)\, (\eta - \eta_o)^k}{k!} \quad . \qquad [19]$$

The coefficients $f^{(k)}_{(\eta_o)}$ were computed from equation 18, and $f(\eta)$ was evaluated by equation 19. These computations were made on the IBM 650.[4/]

Series 19 will have, in general, a finite radius of convergence. To extend the range of validity, analytic continuation was used (4). In this way $f(\eta)$ was delineated for $\eta \geq 0$. Since $U(\zeta, \tau) = f(\eta)$, equation 11 becomes:

$$\theta = \frac{1}{\beta} \ln f(\eta), \quad \eta \geq 0 \quad . \qquad [20]$$

The flux J at the boundary x = 0, is given by:

$$\left[J\right]_{x=0} = - \left[D(\theta) \frac{\partial \theta}{\partial x}\right]_{x=0} .$$

By the use of equations 7, 17, and 20 this can be written:

$$\left[J\right]_{x=0} = - \frac{\omega}{\beta} \sqrt{\frac{\alpha}{2t}} \quad . \qquad [21]$$

The total quantity of water accumulated inside the given region in time t is given by:

$$Q(t) = - A \int_0^t \left[J\right]_{x=0} dt$$

which in view of equation 21 becomes:

$$Q(t) = - \frac{\omega A}{\beta} \sqrt{2 \alpha t} \quad . \qquad [22]$$

Horizontal infiltration and outflow problems are characterized by $\omega < 0$ and $\omega > 0$ respectively, as can be seen from equations 17 and 21.

Linear Diffusivity.

The analysis in the case when the diffusivity function is linear and of the form

$$D(\theta) = \alpha\theta + \beta \qquad (\alpha > 0, \beta > 0) \qquad [23]$$

where α, β are parameters, is analogous to that used in the exponential case.

4/ The use of this or other patented equipment in this study does not imply approval of the product to the exclusion of others that may also be suitable.

The boundary value problems to be solved here are:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[(\alpha \theta + \beta) \frac{\partial \theta}{\partial x}\right], \quad 0 < x < \infty \; ; \; t > 0, \qquad [24]$$

$$\theta (0, t) = \theta_k, \quad (k = 1, 2), \quad t > 0, \qquad [25]$$

$$\theta (x, 0) = \theta_o, \quad x > 0,$$

$$0 < \theta_2 < \theta_o < \theta_1 \quad . \qquad [26]$$

Again k = 1 characterizes the inflow problem and k = 2 the outflow problem.

Solutions of equation 24 of the form

$$\theta (\zeta, \tau) = f (\eta) \qquad [27]$$

were sought, where $\eta = \frac{\zeta}{\sqrt{2\tau}}$, $\zeta = \frac{x}{L}$, and $\tau = \frac{\alpha t}{L^2}$. Equation 24 becomes

$$\left[f (\eta) + \lambda\right] \cdot f'' (\eta) = - f' (\eta) \left[\eta + f' (\eta)\right], \qquad [28]$$

where $\lambda = \beta / \alpha$. Finally, by making the substitution

$$F (\eta) = f (\eta) + \lambda \qquad [29]$$

the preceding problems take the form:

$$F (\eta) F'' (\eta) = - F'' (\eta) \left[\eta + F'' (\eta) \right], \qquad [30]$$

$$F (0) = \theta_k + \lambda, \qquad [31]$$

$$F (\infty) = \theta_o + \lambda,$$

$$0 < \theta_2 < \theta_o < \theta_1 \quad . \qquad [32]$$

By evaluating the nth derivative of equation 30 at $\eta = \eta_o$ the following relationship can be obtained:

$$F^{(n+2)}(\eta_o) = - \frac{1}{F (\eta_o)}\left[\left\{\eta_o + (n+1) F^{(1)}(\eta_o)\right\} F^{(n+1)}(\eta_o) + n\left\{1 + F^{(2)}(\eta_o)\right\} F^{(n)}(\eta_o) + \sum_{k=2}^{n} \binom{n}{k} \left\{F^{(k+1)}(\eta_o) F^{(n-k+1)}(\eta_o) + F^{(k)}(\eta_o) F^{(n-k+2)}(\eta_o)\right\}\right] \qquad [33]$$

$$n = 2, 3, 4\text{----}$$

The Taylor series solution is of the form:

$$F(\eta) = \sum_{k=0}^{\infty} F^{(k)}_{(\eta_o)} (\eta - \eta_o)^k / k! \quad . \qquad [34]$$

The coefficients $F^{(k)}_{(\eta_o)}$ were computed from equation 33 and $F(\eta)$ was evaluated from equation 34. These computations were made on the IBM 650.

In a manner similar to that used in the exponential diffusivity case, the condition of equation 32 was replaced by:

$$F'(\eta_o) = \omega \quad .$$

The flux, J, at x = 0 and the total quantity of water accumulated in the region are given by:

$$[J]_{x=0} = -\omega(\theta_k + \lambda)\sqrt{\frac{\alpha}{2t}} \qquad [35]$$

and

$$Q(t) = -\omega(\theta_k + \lambda) A \sqrt{2\alpha t} \quad . \qquad [36]$$

Again, $\omega < 0$ denotes the inflow problem and $\omega > 0$ the outflow problem.

Method of Computation.

The procedures for obtaining the power series solutions of equations 8 and 24 for the exponential and linear diffusivity functions were similar. Flow diagrams outlining the computational program are given in Appendix III. The program consisted of the following: (a) computation of the Taylor series coefficients $f^{(k)}(\eta_o)$ or $F^{(k)}(\eta_o)$, (b) evaluation of the Taylor series for $f(\eta)$ or $F(\eta)$ in its interval of convergence. The interval of convergence was found to be finite. This necessitated extending the power series solution to a larger interval of convergence by the process of analytic continuation (4). When oscillations in the values of $f(\eta)$ or $F(\eta)$ were detected,[5/] a value of $\eta = \eta_b$ just to the left of the right endpoint of the interval of convergence was selected. The function $f(\eta)$ (or $F(\eta)$) and a sufficient number of its successive derivatives were computed at η_1, which determined a Taylor series solution defined over an interval of convergence $\eta_1 \leq \eta \leq \eta_2$. The above computational procedure was continued until the difference between successive ordinates was less than some preassigned quantity.

A particular pair of values of θ_k and ω were selected and the computational procedure described above was carried out. Corresponding to the θ_k and ω chosen, a particular value of θ_o was obtained. By varying ω with a fixed

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5/ A series consisting of 24 terms was used to evaluate $f(\eta)$ or $F(\eta)$.

θ_k a series of problems was solved. These computed results are shown in figures 1, 2, 3, and 4. The date in all the other figures in this report are based on these computed results as explained below.

Results - Exponential Diffusivity.

The water content, θ, is related to f (η) by the equivalent equations

$$f(\eta) = e^{\beta\theta} \text{ or } \ln f(\eta) = \beta\theta \quad . \qquad [37]$$

In order to use the computed results in practice, transformations must be made into measurable quantities. For that purpose, the following water content ratios are convenient to use:

$$\frac{\theta - \theta_2}{\theta_o - \theta_2} = \frac{\ln f(\eta) - \ln f(0)}{\ln f(\infty) - \ln f(0)} \text{ , outflow} \qquad [38]$$

$$\frac{\theta - \theta_o}{\theta - \theta_o} = \frac{\ln f(\eta) - \ln f(\infty)}{\ln f(0) - \ln f(\infty)} \text{ , inflow} \qquad [39]$$

where $f(0) = e^{\beta\theta_2}$ in the outflow case, $f(0) = e^{\beta\theta_1}$ in the inflow case, and $f(\infty) = e^{\beta\theta_o}$ in both cases. In this way these ratios will have values in the range from 0 to 1.0. From the results of the computations of f (η) for various values of the parameter ω, ln f (η) was computed for these same values of the parameters and the ratios in 38 and 39 were obtained. It was more convenient to graph $(\theta - \theta_2)/(\theta_o - \theta_2)$ as a function of the ratio

$$x / \sqrt{D_2 t} = \eta \sqrt{2 / e^{\beta\theta_2}} \qquad [40]$$

and $(\theta - \theta_o)/(\theta_1 - \theta_o)$ as a function of the ratio

$$x / \sqrt{D_o t} = \eta \sqrt{2 / e^{\beta\theta_o}} \quad . \qquad [41]$$

Figures 5, 6, and 7 show the plots for inflow and outflow, respectively, for various values of the parameter $\omega = f'(0)$.

The procedure for translating the dimensionless quantities of figures 5, 6, and 7 into the corresponding specific dimensional quantities of a given problem is as follows: For a given porous medium the diffusivity function must be known. The initial water content, θ_o, and the water content at x = 0, θ_k must also be given. When k = 1, i.e. infiltration, the ratio D_1/D_o is equal to $e^{\beta(\theta_1 - \theta_o)}$ where $\theta_1 > \theta_o$. When k = 2, i.e. outflow, the ratio D_o/D_2 is given by $e^{\beta(\theta_o - \theta_2)}$ where $\theta_o > \theta_2$. Since the diffusivity function is assumed to be exponential and known, both α and β are known.

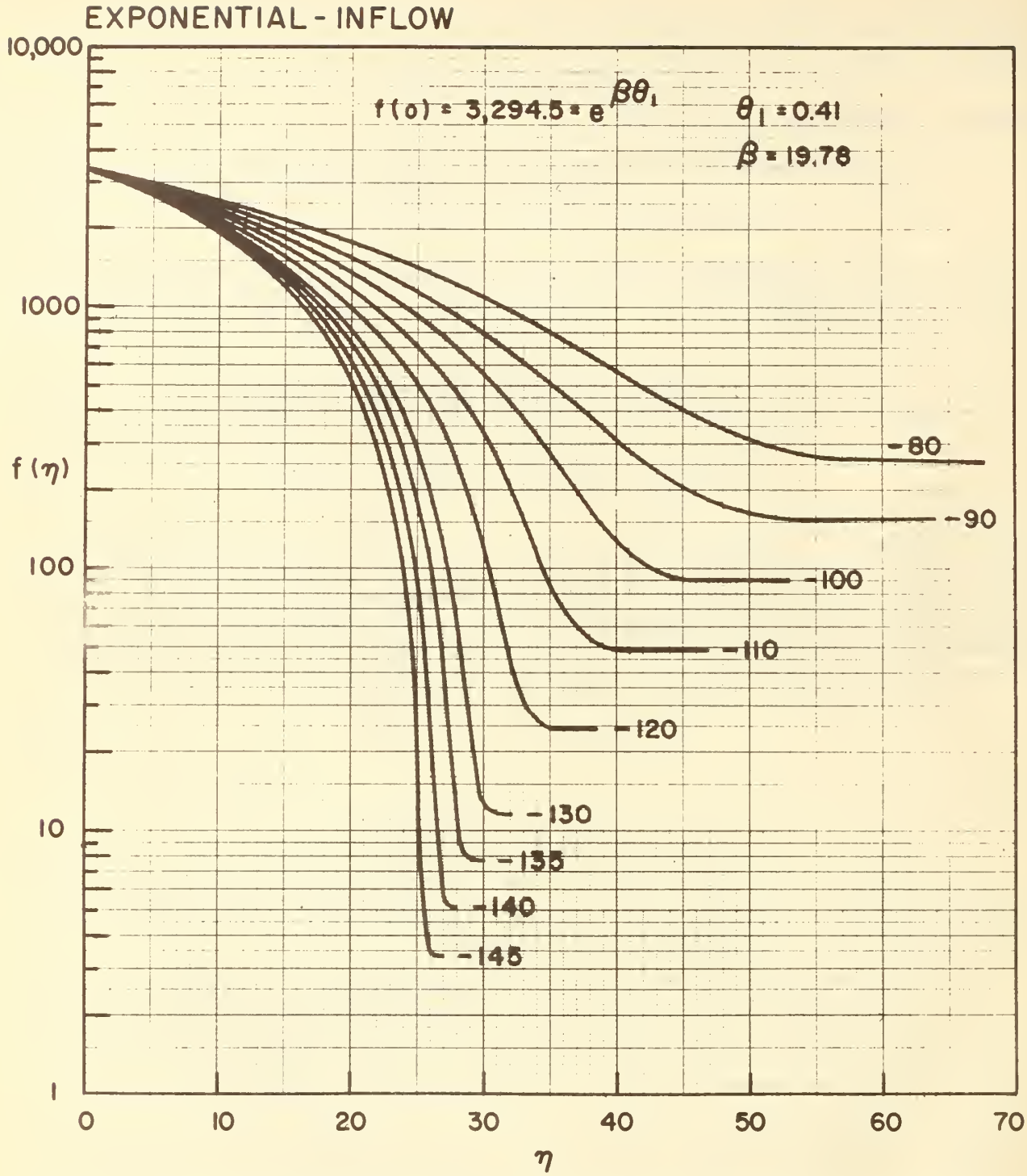

FIGURE 1.--Original computed results for inflow with exponential diffusivity. The parameter on the curves is $\omega = f'(0)$.

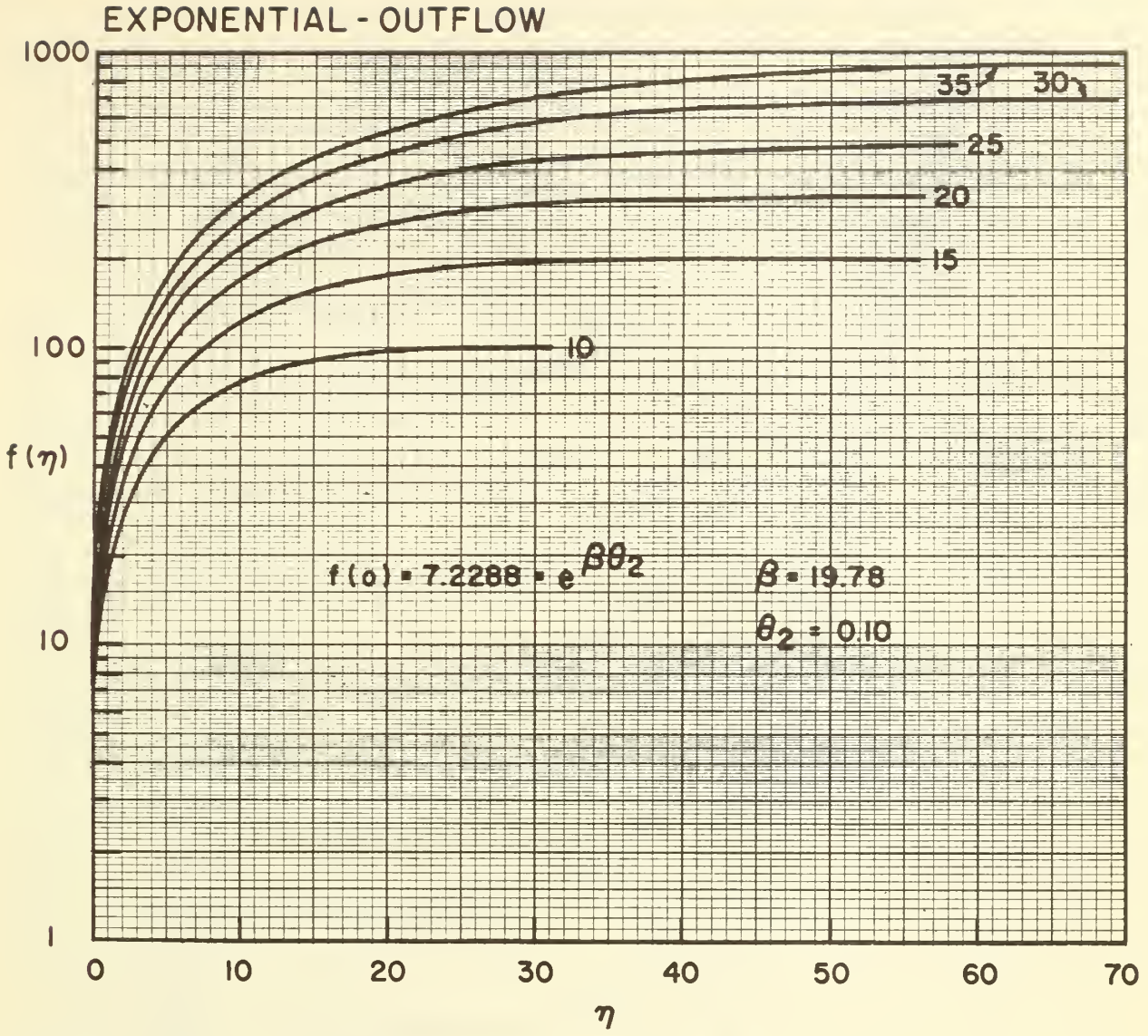

FIGURE 2.--Original computed results for outflow with exponential diffusivity. The parameter on the curves is $\omega = f'(0)$.

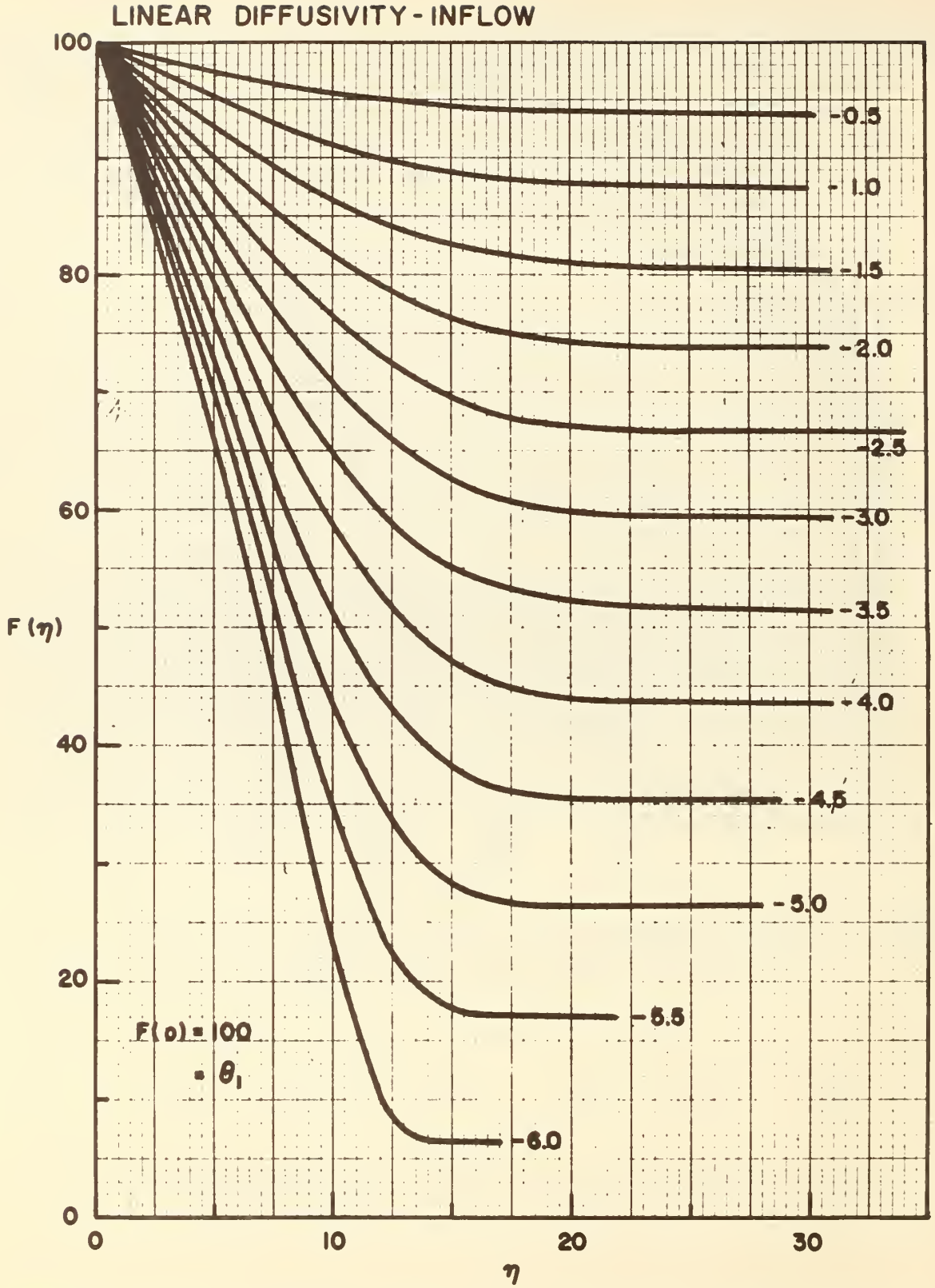

FIGURE 3.--Original computed results for inflow with a linear diffusivity. The parameter on the curves is $\omega = F'(0)$.

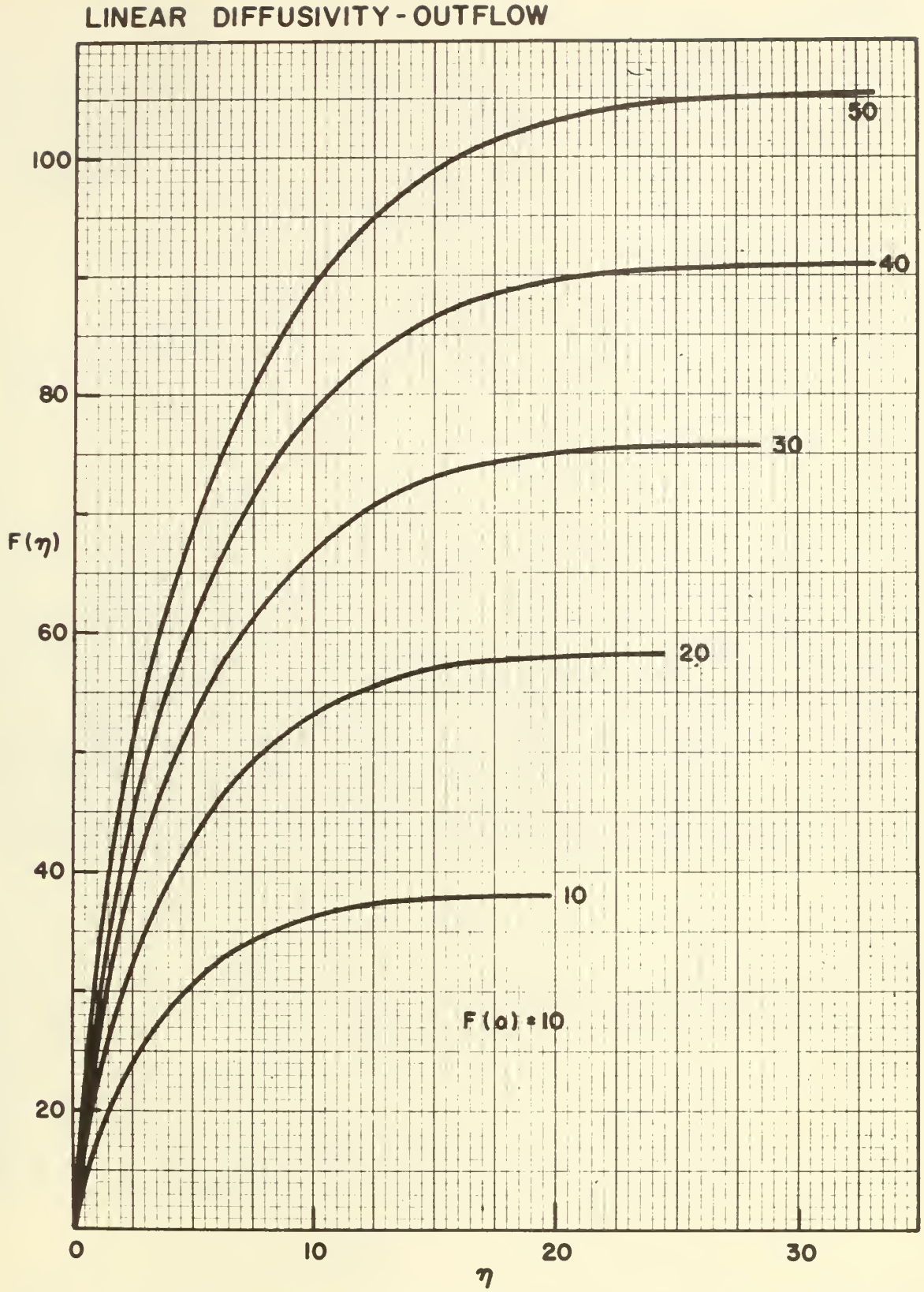

FIGURE 4.--Original computed results for outflow with a linear diffusivity. The parameter on the curves is $\omega = f'(0)$.

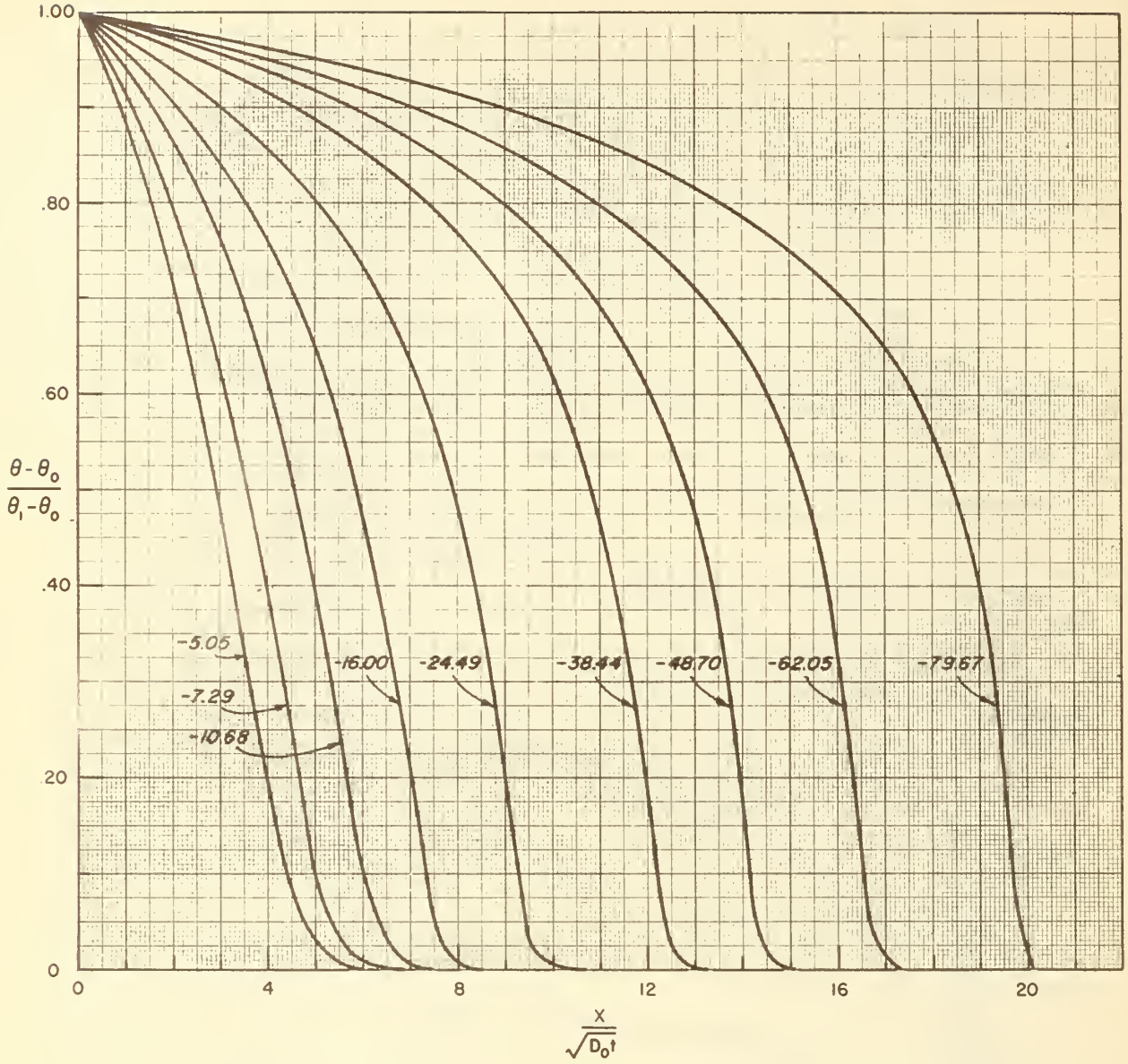

FIGURE 5.--Dimensionless plot of water content versus $\frac{x}{\sqrt{D_o t}}$ for inflow with an exponential diffusivity. The parameter is ω^*.

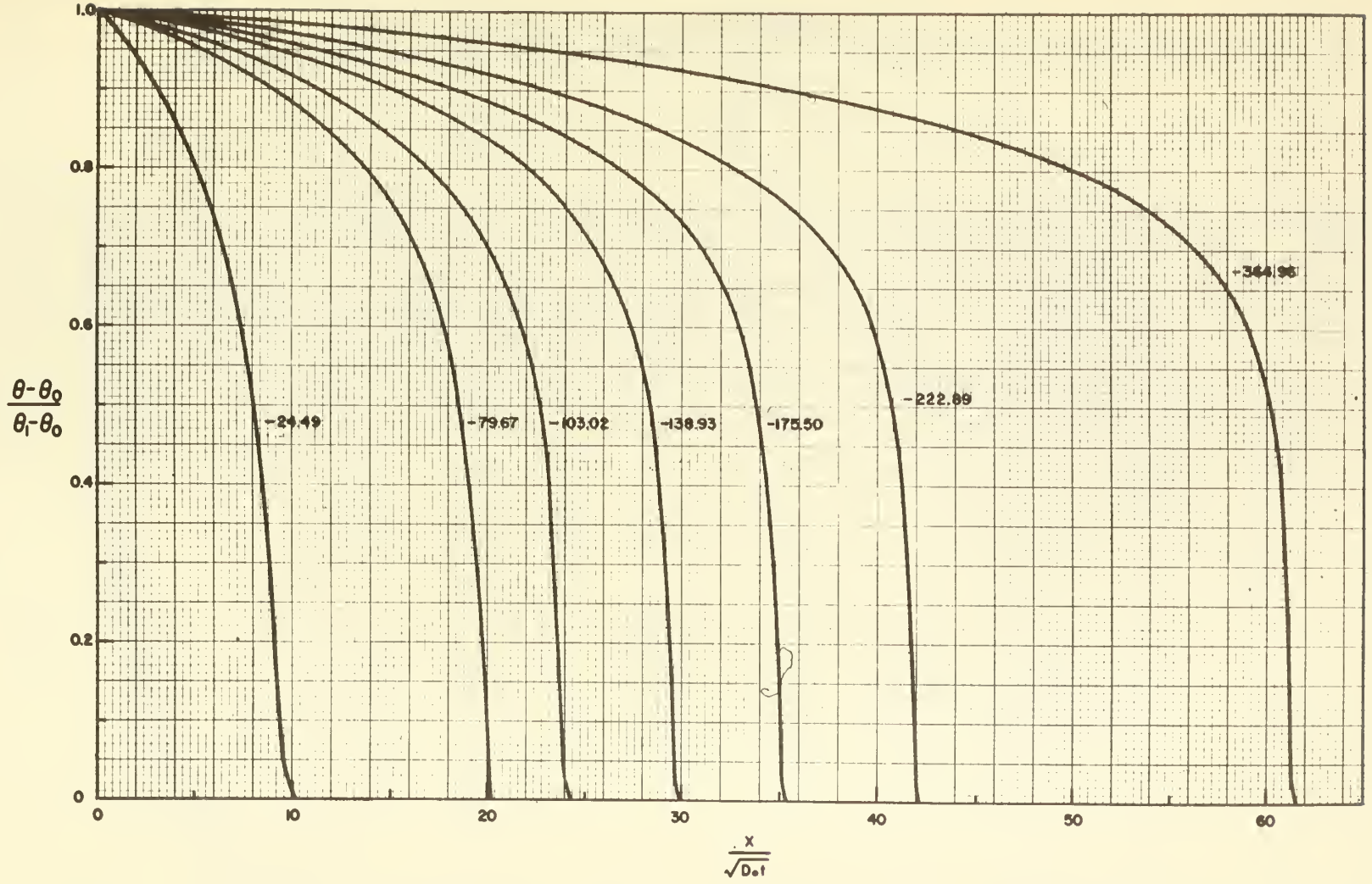

FIGURE 6.--Dimensionless plot of water content versus $\frac{x}{\sqrt{D_o t}}$ for inflow with an exponential diffusivity. The parameter is ω^*.

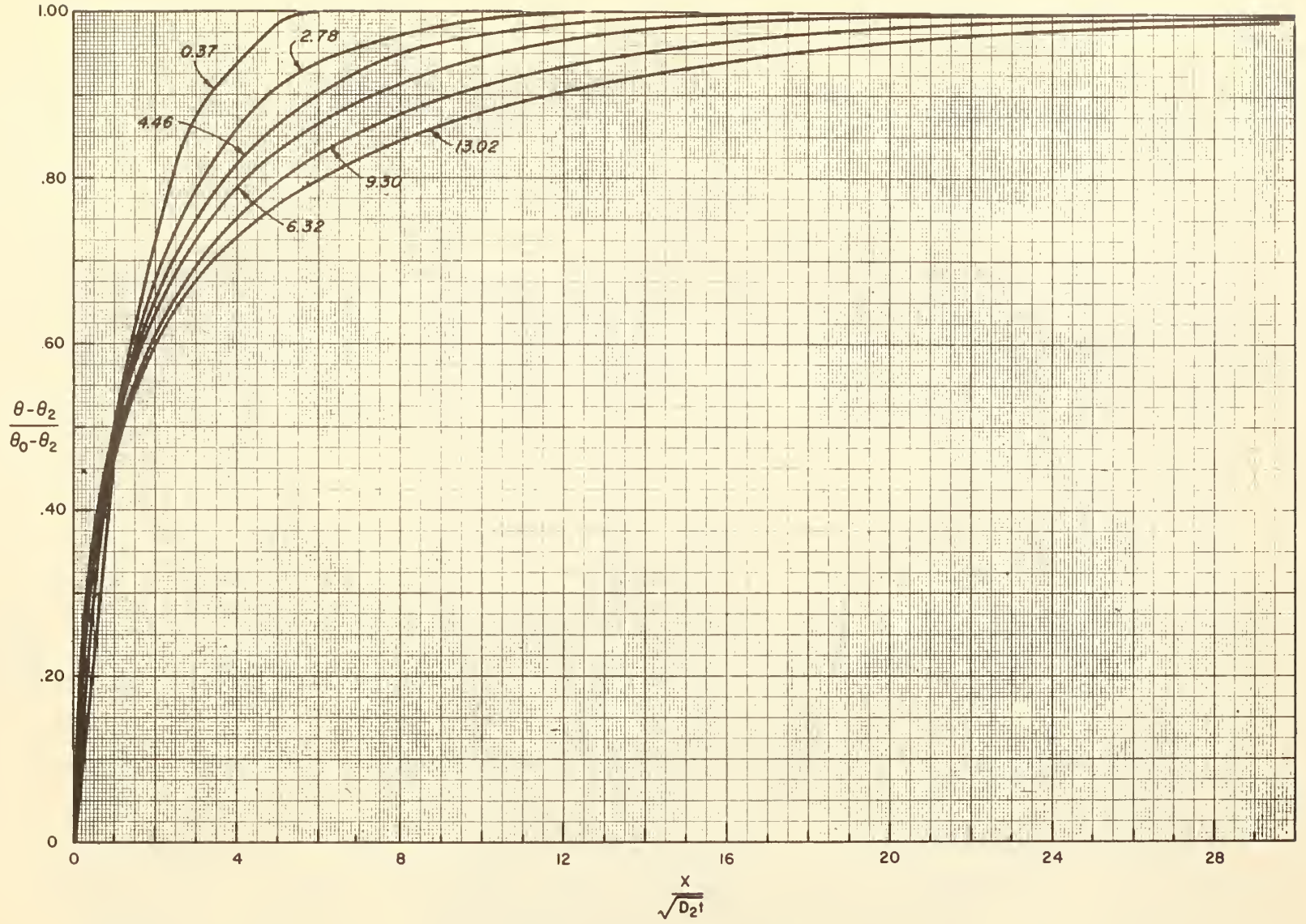

FIGURE 7.--Dimensionless plot of water content versus $\frac{x}{\sqrt{D_2 t}}$ for inflow with an exponential diffusivity. The parameter is ω^*.

To determine the value of ω corresponding to given values of $\theta_o \neq 0$ and θ_1 the following procedure is used in connection with the exponential diffusivity inflow case. Compute the ratio D_1/D_o from $e^{\beta\theta_1}/e^{\beta\theta_o}$. Find a value of ω^* corresponding to this diffusivity ratio by use of figure 8. Calculate ω from:

$$\omega = \omega^* \sqrt{e^{\beta\theta_o}} \quad . \qquad [42]$$

This equation is derived in Appendix I. The construction of the graph in figure 8 is discussed in Appendix II.

The outflow case is treated in a similar manner. The ratio D_o/D_2 is computed from $e^{\beta\theta_o}/e^{\beta\theta_2}$ and a value of ω^* found by the use of figure 8. The appropriate value of ω corresponding to the given θ_2 is then obtained from:

$$\omega = \omega^* \sqrt{e^{\beta\theta_2}} \quad . \qquad [43]$$

The flux and accumulated water content can now be found from equations 21 and 22.

The magnitude of the ratio of the outflow flux to the inflow flux when (a) $|\theta_k - \theta_o|$ is the same for outflow as for inflow, (b) when θ_2 for outflow is equal to θ_o for inflow, and (c) when θ_1 for inflow is equal to θ_o for outflow is given by:

$$\left|\frac{[J]\ \text{out}}{[J]\ \text{in}}\right|_{x=0} = \frac{\frac{\omega\ \text{out}}{\beta}\sqrt{\frac{\alpha}{2t}}}{\frac{\omega\ \text{in}}{\beta}\sqrt{\frac{\alpha}{2t}}}$$

$$= \left|\frac{\omega\ \text{out}}{\omega\ \text{in}}\right| = \left|\frac{\omega^*\ \text{out}}{\omega^*\ \text{in}}\right| \quad .$$

Under the above conditions $D_o/D_2 = D_1/D_o$. From figure 8 and by assuming the above conditions, note that the inflow flux is larger in magnitude than the outflow flux.

The results of the computations, as shown in figures 5, 6, and 7, could be presented with a diffusivity ratio as the parameter, as Gardner (1) has done. However, since the computations were originally made with ω and θ_k as the boundary conditions it was convenient to use ω^* as the parameter in the curves of figures 5, 6, and 7. Also, if the flux is desired, the quantity ω is needed.

Results - Linear Diffusivity.

The water content is related to F (η) by

$$F(\eta) = \theta + \lambda \, , \qquad [44]$$

where θ is defined in 27 and $\lambda = \beta/\alpha$. The graphs for the inflow and outflow cases are shown in figures 9 and 10, respectively, for the specific case $\lambda = 0$. It was convenient to plot the ratio θ/θ_1 versus $x/\sqrt{2D_1 t}$ in the inflow case

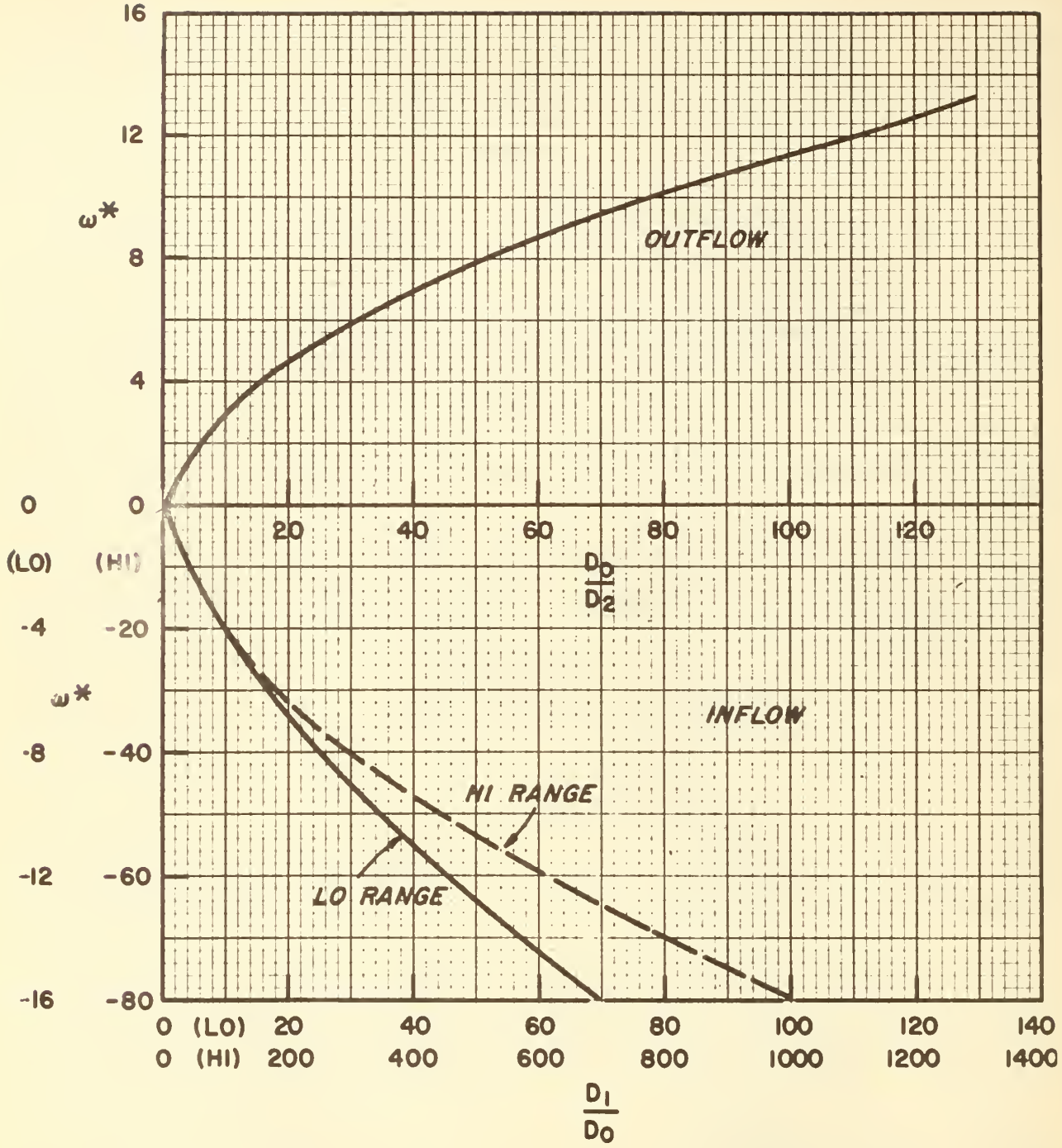

FIGURE 8.--Graph of diffusivity ratio versus ω*. The construction of this graph is explained in Appendix II.

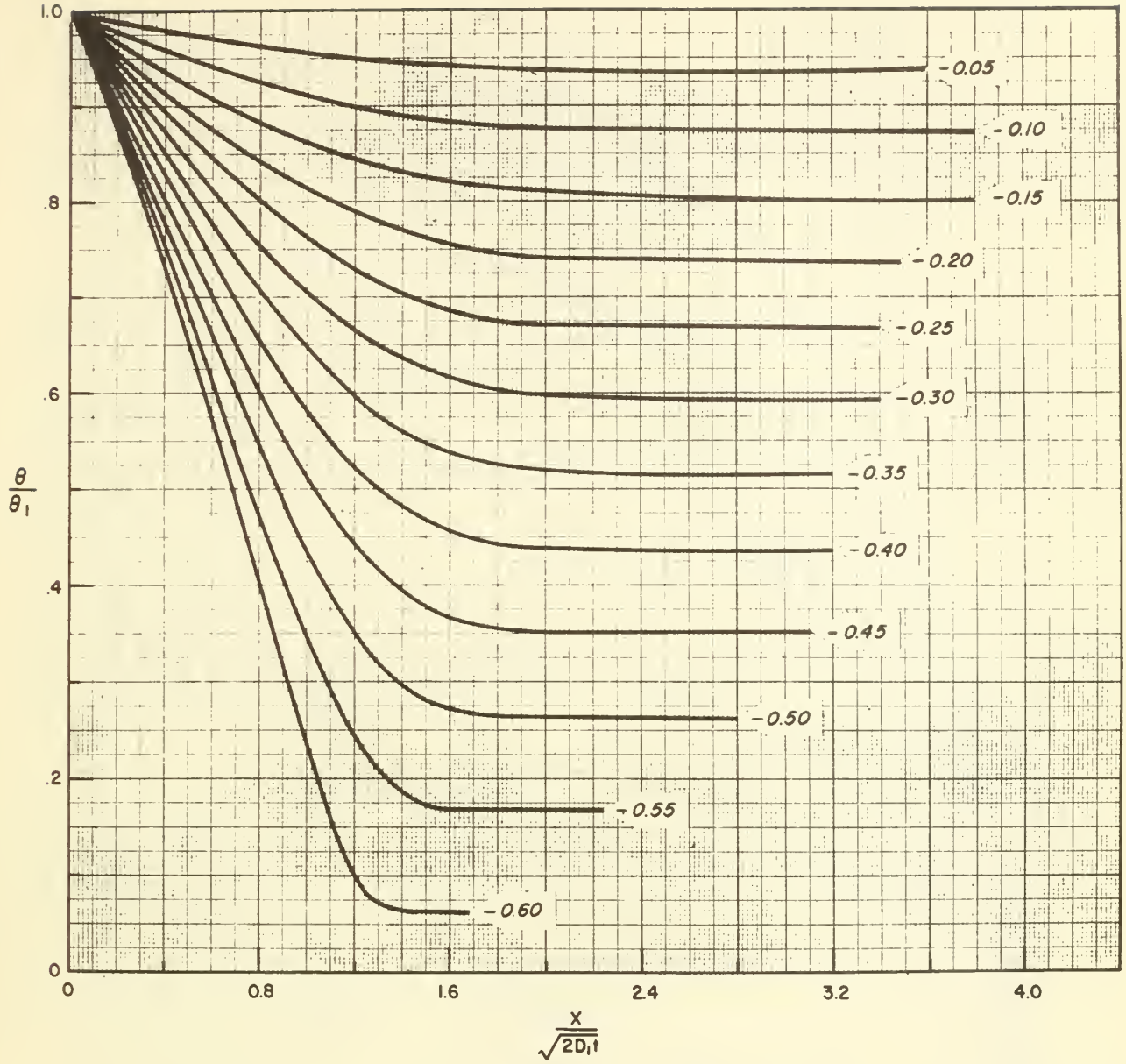

FIGURE 9.--Dimensionless plot of water content versus $x/\sqrt{2D_1 t}$ for inflow with a linear diffusivity function in which $\lambda = 0$ (see equations 29 and 23). The parameter is ω^*.

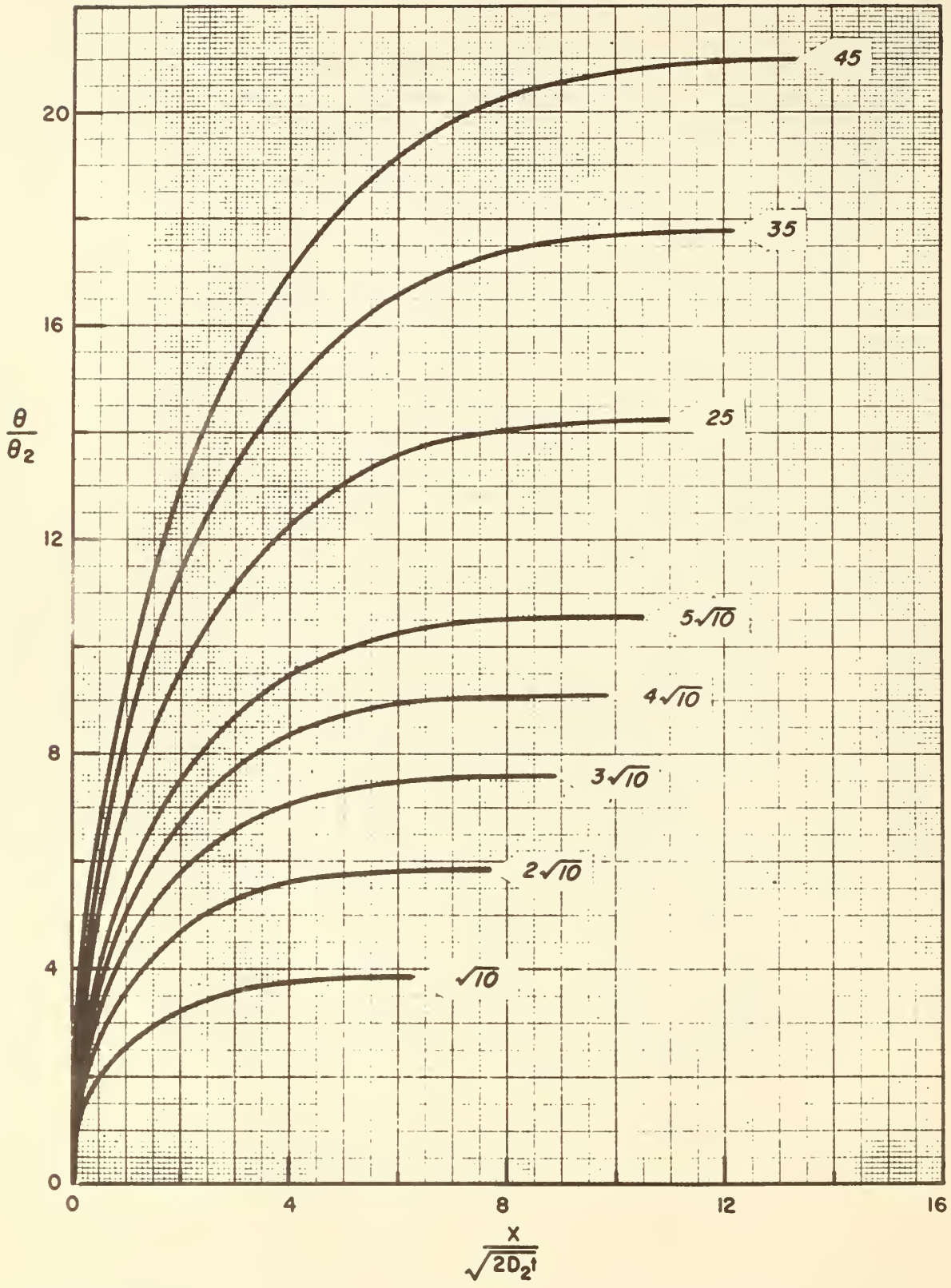

FIGURE 10.--Dimensionless plot of water content versus $x/\sqrt{2D_2t}$ for outflow with a linear diffusivity function in which $\lambda = 0$. The parameter is ω^*.

and the ratio θ/θ_2 versus $x/\sqrt{2D_2t}$ in the outflow case. In each graph the curves are labeled with ω as a parameter.

The procedure for obtaining the appropriate value of ω for a given pair of values of θ_o and θ_k which define a specific flow problem is as follows. Calculate D_o and D_k by the use of equation 23. Form the ratio D_o/D_2 of D_1/D_o and use figure 11 to obtain ω^*. This assumes $\lambda = 0$. Calculate ω from

$$\omega = \omega^* \sqrt{\theta_k} \quad .$$

Next select the appropriate curve in figures 9 or 10 which is labeled with the above value of ω.

The flux and accumulated quantity of water can be calculated from equations 35 and 36.

The magnitude of the ratio of inflow flux to outflow flux is given by:

$$\left|\frac{[J]\ \text{in}}{[J]\ \text{out}}\right| x = 0 = \left|\frac{\omega\ \text{in}\ \ (\theta_1 + \lambda)}{\omega\ \text{out}\ (\theta_2 + \lambda)}\right| \quad . \qquad [45]$$

If we assume (a) that θ_1 for inflow is equal to θ_o for outflow and (b) that θ_2 for outflow is equal to θ_o for inflow, then from the equation for the diffusivity, equation 23, it follows that

$$\left(\frac{D_1}{D_o}\right)_{IN} = \left(\frac{D_o}{D_2}\right)_{OUT} \qquad (D_o)\ IN \neq (D_o)\ OUT \quad .$$

Inspection of figure 11 shows that for a given diffusivity ratio $|\omega\ \text{in}|$ is less than $|\omega\ \text{out}|$.

If $\lambda = 0$ then:

$$\left|\frac{J\ \text{in}}{J\ \text{out}}\right| x = 0 = \left|\frac{\omega\ \text{in}\ \ \theta_1}{\omega\ \text{out}\ \theta_2}\right|$$

$$= \left|\frac{\omega^*\ \text{in}}{\omega^*\ \text{out}} \left(\frac{\theta_1}{\theta_2}\right)^{3/2}\right|$$

$$= \left|\frac{\omega^*\ \text{in}}{\omega^*\ \text{out}} \left(\frac{D_1}{D_o}\right)^{3/2}_{\text{in}}\right|$$

$$= \left|\frac{\omega^*\ \text{in}}{\omega^*\ \text{out}} \left(\frac{D_o}{D_2}\right)^{3/2}_{\text{out}}\right| \quad .$$

The magnitude of this ratio can be computed with the aid of figure 11. The computations show that the inflow flux is larger in magnitude than the outflow flux. Some computed results are shown in Table 1.

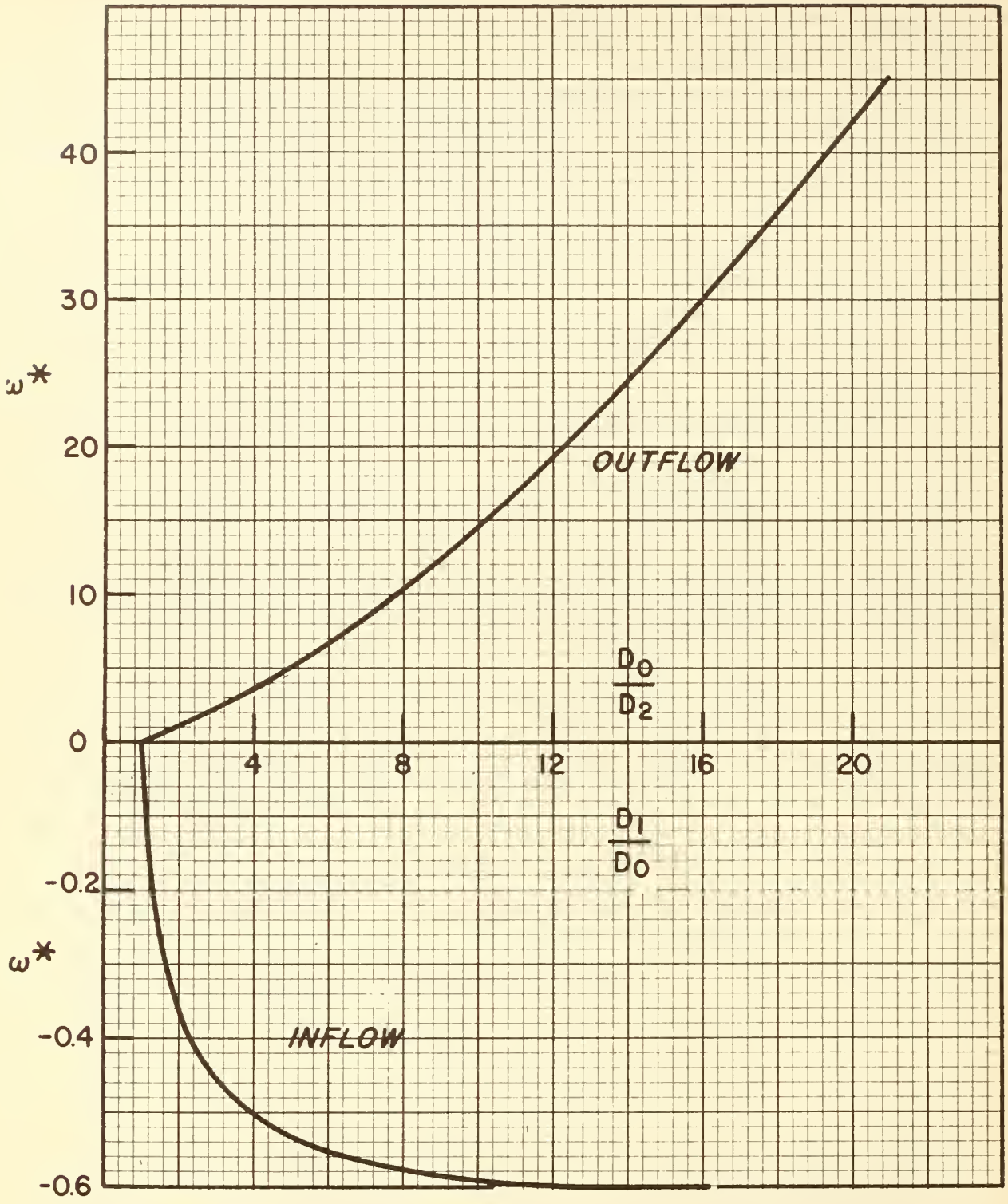

FIGURE 11.--Graph of diffusivity ratio versus ω*. The construction of this graph is explained in Appendix II.

TABLE 1.

$\frac{D_1}{D_o}$ in or $\frac{D_o}{D_2}$ out	$\frac{J \text{ in}}{J \text{ out}}$
4	1.12
8	1.24
12	1.29

Applications.

If a flow system can be described by equation 1 and if the boundary conditions stated in equations 9 and 10 are applicable, then it is possible to predict the water content as a function of time and position in the flow system. The flux and the total amount of water accumulated can also be calculated. If the diffusivity is assumed to be exponential or linear the constants in the diffusivity function can be obtained from a measurement of the water content distribution at a given time, the appropriate computed distribution of water content as shown in figures 5, 6, 7, 9, or 10, and the value of the diffusivity at one water content.

The diffusivity data for two soils, Sarpy loam and Geary silt loam, appear to be of exponential nature. We have determined the water content as a function of distance for several different boundary conditions as shown in figures 12 and 13. For Sarpy loam for infiltration with $\theta_o = 0.061$ and $\theta_1 = 0.41$ the ratio D_1/D_o is: $D_1/D_o = e^{\beta(\theta_1 - \theta_o)} = e^{19.78\,(0.349)} = 992.$

From figure 8, for inflow, ω^* is seen to be about -79.5. By using equation 42 ω is calculated to be -145. The distributions of water content shown in figure 12 were calculated from the curves of figure 5 labeled with $\omega^* = -79.67$. The other curves in figure 12 were obtained in a similar fashion.

The flux at $x = 0$ for Sarpy loam with $\theta_1 = 0.41$ and $\theta_o = 0.061$ is given by equation 21:

$$[J]_{x=0} = -\frac{-144}{19.78}\sqrt{\frac{1.24 \times 10^{-4}}{2t}}$$

$$= \frac{5.73 \times 10^{-2}}{\sqrt{t}} \qquad \text{(cm/sec)} \quad .$$

The total quantity of water accumulated in time t is given by equation 22:

$$Q(t) = -\frac{-144\,A}{19.78}\sqrt{2 \times 1.24 \times 10^{-4} \times t}$$

$$= 0.115\,A\sqrt{t} \qquad (\text{cm}^3) \quad .$$

Figure 13 shows θ versus x for indicated values of t for outflow. These curves were constructed in a manner similar to that described above.

Note that, for the exponential diffusivity case, the water content scale can be translated up or down without influencing the shape of the (θ vs x) curve. That is, the data of figure 13 could apply to any boundary conditions where $(\theta_1 - \theta_o)$ is constant. For example, the data for Sarpy loam with

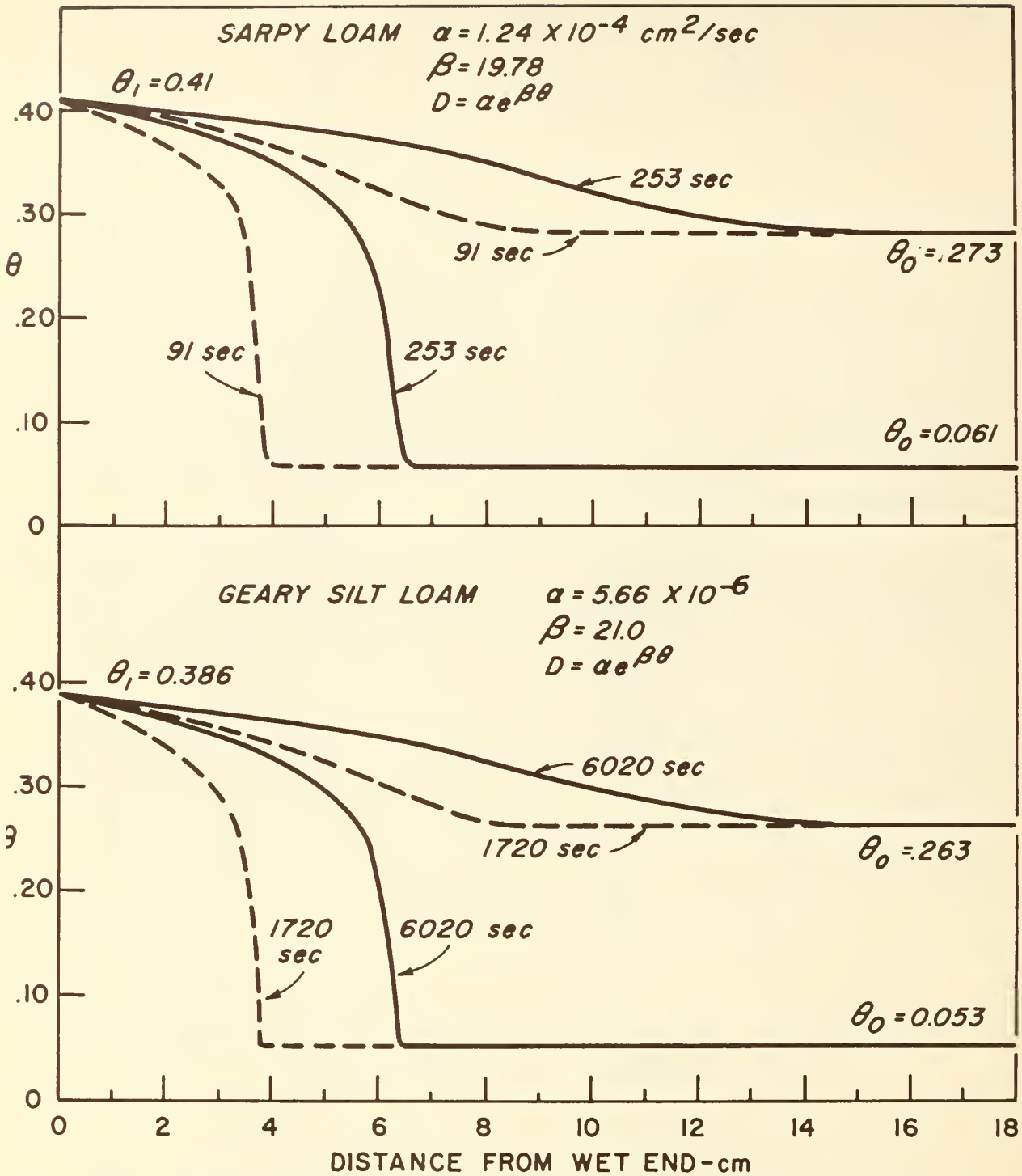

FIGURE 12.--Water content versus distance for horizontal inflow for Geary silt loam and Sarpy loam with boundary conditions as indicated.

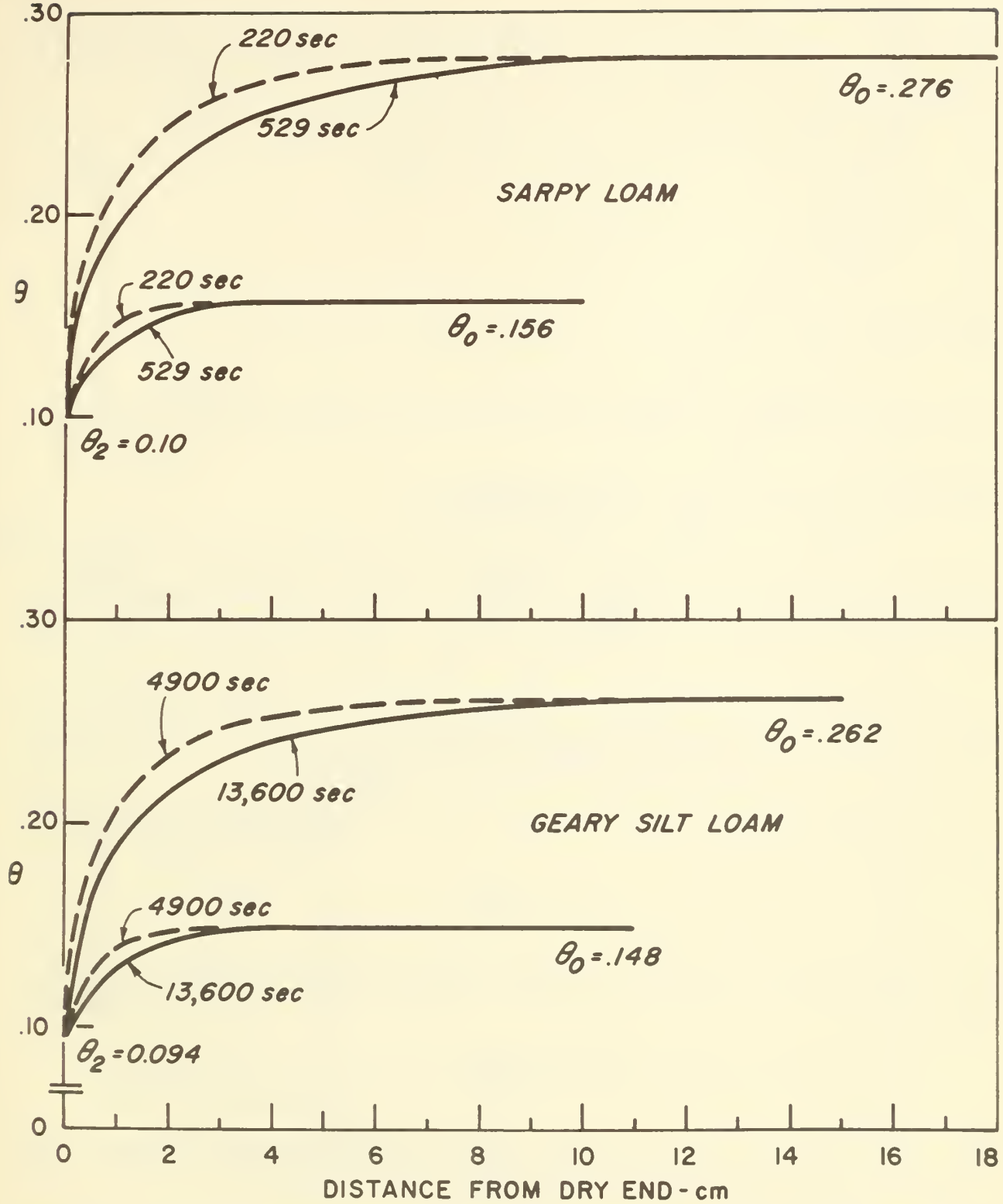

FIGURE 13.--Water content versus distance for horizontal outflow for Geary silt loam and Sarpy loam with boundary conditions as indicated.

$\theta_1 = 0.41$ and $\theta_o = 0.061$ would also apply to the case with $\theta_1 = 0.460$ and $\theta_o = 0.111$. The ratio $\frac{D_1}{D_o}$ would be the same but, of course, the flux would be changed, because ω would be changed. The value of ω corresponding to $\theta_1 = 0.46$ and $\theta_o = 0.111$ is

$$\omega = \omega^* \sqrt{e^{19.78 \cdot 0.111}}$$

$$= -79 \times 3.03$$

$$= -239$$

This value of ω would be used to compute the flux for these new boundary conditions.

Similar application could be made of the linear data if the linear diffusivity function is known and if the boundary conditions agree with those of equations 25 and 26.

Comparison of Linear and Exponential Cases.

In order to make a comparison between an outflow system in which a linear diffusivity function applies and one in which an exponential function applies, we will impose the conditions that (a) for $\theta = \theta_2$ the exponential and linear diffusivities are the same and (b) for $\theta = \theta_o$ the exponential and linear diffusivities are equal. These conditions are shown graphically in figure 14.

The water content as a function of $\frac{x}{\sqrt{2D_2 t}}$ is shown in figure 15 for two examples where θ_2 has been set equal to 0.01 and θ_o equal to 0.106 and 0.21. It is evident from this figure that the exponential case does not drain as rapidly as the linear case under the imposed conditions.

The outflow flux for the linear case is given by application of equation 35 to be:

$$J = 0.318 \sqrt{\frac{D_2}{t}}$$

where the appropriate ω and α values have been determined from the diffusivity ratio and θ_2. The outflow flux for the exponential case is given by applying equation 21:

$$J = 0.223 \sqrt{\frac{D_2}{t}}$$

where ω, β and α have been determined from the diffusivity ratio and θ_2. From these two equations it is seen that at any given t the flux for the linear case is larger than that for the exponential case.

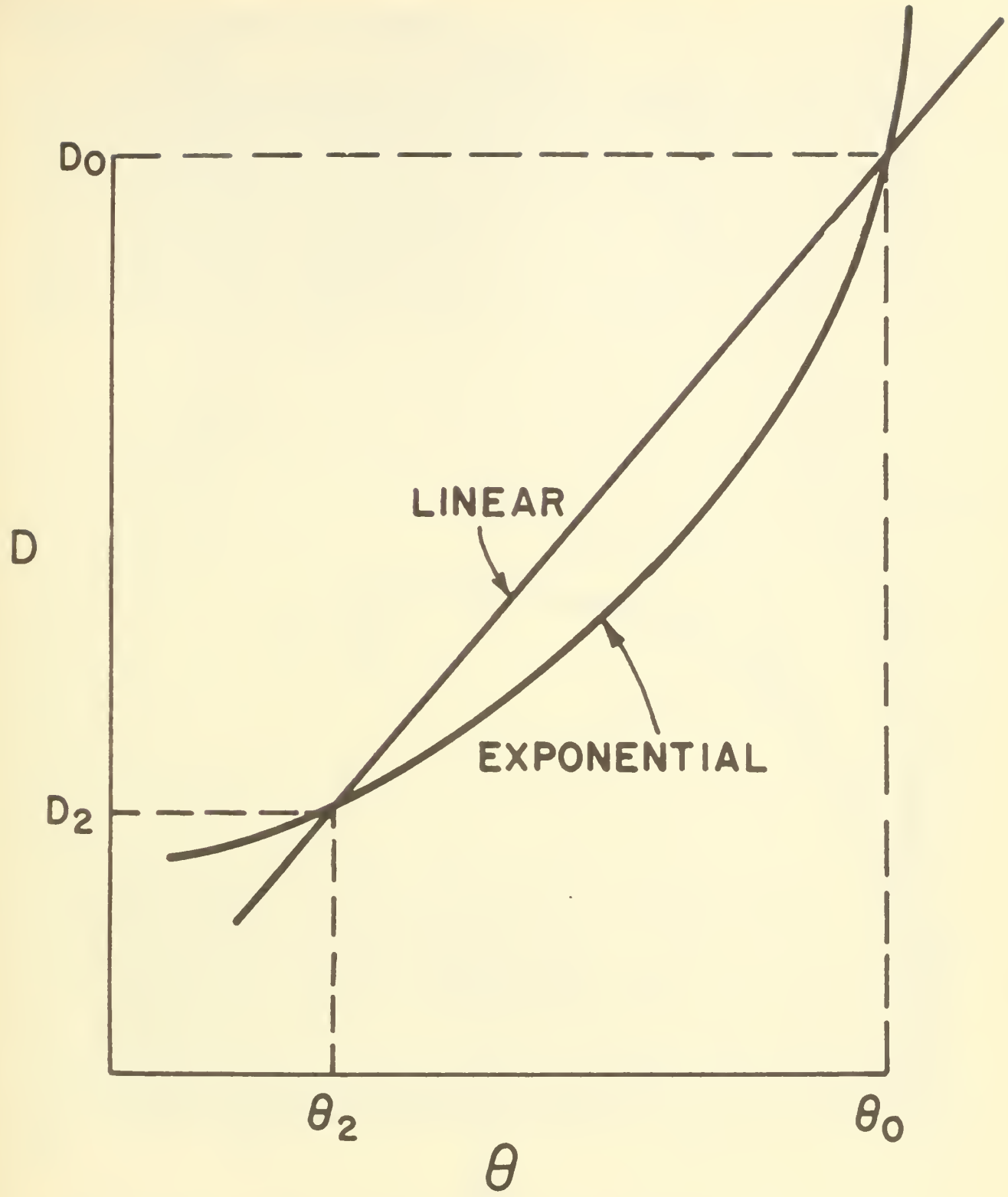

FIGURE 14.--Graph of D versus θ to show how the comparison of exponential and linear outflow cases shown in figure 15 was made.

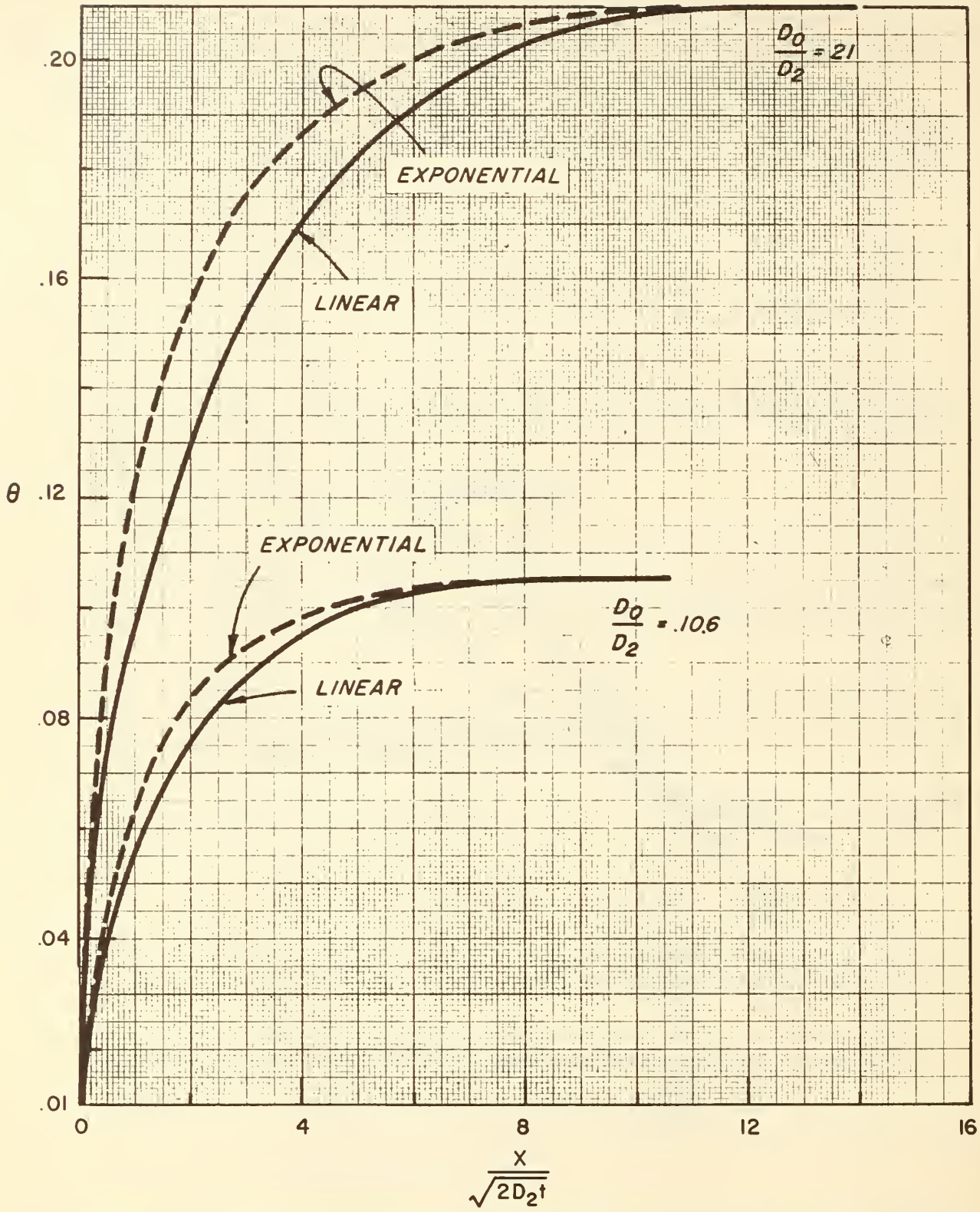

FIGURE 15.--Comparison of linear and exponential outflow cases.

Estimation of Diffusivity Functions.

If it is assumed that the diffusivity function is of either the exponential or linear type and if the value of the diffusivity at the water content θ_o is known, the constant in the diffusivity function can be determined from a measurement of the water content distribution at a given time in either an inflow or outflow system. By assuming for the moment that the diffusivity function is exponential, that D_o is known and that measurements are made on an inflow system, the resulting θ versus x data could be plotted in the reduced form of figure 1. By matching curves and interpolation a value of ω could be estimated and from this and θ_o, θ_1, and D_o a value of D_1 could be obtained. Since D_o and D_1 are now known, β and α can be determined. A similar procedure could be followed if a linear diffusivity is assumed.

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Derivation of Slope Conversion Formulas for the Exponential and Linear Diffusivity Functions

(A) Inflow Case for the Exponential Function $f(\eta) = e^{\beta\theta}$ $(\beta > 0)$. $= X/\sqrt{2\alpha t}$.

The plot of $(\theta - \theta_o)/(\theta_1 - \theta_o)$ versus $x/\sqrt{D_o t}$ gives a single curve for all boundary conditions, where $(\theta_1 - \theta_o)$ = const. or D_1/D_o = const. Also, $\eta = \zeta\sqrt{D_o/(2\alpha)}$, where $\zeta = x/\sqrt{D_o t}$.

Now,

$$\omega = \left[\frac{df(\eta)}{d\eta}\right]_{\eta=0} = \left[\frac{df(\eta)}{d\theta}\frac{d\theta}{d\zeta}\frac{d\zeta}{d\eta}\right]_{\eta=0} = \left[\frac{\beta e^{\beta\theta}}{\sqrt{D_o/(2\quad)}}\frac{d\theta}{d\zeta}\right]_{\zeta=0}$$

$$= \sqrt{2}\,\beta e^{\beta\left(\theta_1 - \frac{\theta_o}{2}\right)}\left[\frac{d\theta}{d\zeta}\right]_{\zeta=0}$$

since $D_o = \alpha e^{\beta\theta_o}$.

Similarly, for $f^*(\eta) = e^{\beta\theta^*}$,

$$\omega^* = \left[\frac{df^*(\eta)}{d\eta}\right]_{\eta=0} = \sqrt{2}\,\beta e^{\beta\left(\theta_1^* - \frac{\theta_o^*}{2}\right)}\left[\frac{d\theta^*}{d\zeta}\right]_{\zeta=0}$$

since $D_o^* = \alpha e^{\beta\theta_o^*}$.

Moreover, for two different boundary conditions such that

$$\theta_1^* - \theta_o^* = \theta_1 - \theta_o\,, \qquad [1]$$

the two curves corresponding to ω and ω^* plot on one curve in the graph of $(\theta - \theta_o)/(\theta_1 - \theta_o)$ versus ζ. Therefore,

$$\left[\frac{d\theta}{d\zeta}\right]_{\zeta=0} = \left[\frac{d\theta^*}{d\zeta}\right]_{\zeta=0}\,. \qquad [2]$$

Forming the ratio of ω^* to ω and making use of conditions as shown in equations 1 and 2, the following result obtains

$$\omega^* = \omega\sqrt{e^{\beta(\theta_o^* - \theta_o)}}\,. \qquad [3]$$

If $\theta_o^* = 0$ as a standard, then equation 3 becomes

$$\omega = \omega^*\sqrt{e^{\beta\theta_o}}\,. \qquad [4]$$

(B) Outflow Case for the Exponential Function.

Just as in the inflow case, the following expressions are found:

$$\omega^* = \sqrt{2}\,\beta\left(e^{\beta\theta_2^*}/\sqrt{D_2^*/\alpha}\right)\left[\frac{d\theta^*}{d\zeta}\right]_{\zeta=0}$$

$$\omega = \sqrt{2}\,\beta\left(e^{\beta\theta_2}/\sqrt{D_2/\alpha}\right)\left[\frac{d\theta}{d\zeta}\right]_{\zeta=0} .$$

By making use of the relations

$$D_2^* = \alpha e^{\beta\theta_2^*}, \quad D_2 = \alpha e^{\beta\theta_2},$$

$$\left[\frac{d\theta^*}{d\zeta}\right]_{\zeta=0} = \left[\frac{d\theta}{d\zeta}\right]_{\zeta=0}$$

results in the formula

$$\omega^* = \omega\sqrt{e^{\beta(\theta_2^* - \theta_2)}} . \qquad [5]$$

For $\theta_2^* = 0$, equation 5 becomes

$$\omega = \omega^*\sqrt{e^{\beta\theta_2}} . \qquad [6]$$

(C) Inflow Case for the Linear Diffusivity Function $D = \alpha\theta + \beta$, ($\alpha > 0$, $\beta > 0$), $\eta = x/\sqrt{2\alpha t}$.

By definition, $F(\eta) = \theta + \lambda$, where $\lambda = \beta/\alpha$ and $D_1 = \alpha\theta_1 + \beta$. Also, $F(0) = \theta_1 + \lambda$ and $\eta = \zeta\sqrt{D_1/\alpha}$, where $\zeta = x/\sqrt{2D_1 t}$.

Now,

$$\omega = \left[\frac{dF(\eta)}{d\eta}\right]_{\eta=0} = \left[\frac{d\zeta}{d\eta}\,\frac{dF}{d\zeta}\right]_{\zeta=0}$$

$$= \frac{1}{\sqrt{D_1/\alpha}}\left[\frac{dF}{d\zeta}\right]_{\zeta=0} = \sqrt{F(0)}\left[\frac{d}{d\zeta}\left(\frac{F(\eta)}{F(0)}\right)\right]_{\zeta=0}$$

$$= \sqrt{F(0)}\left[\frac{d}{d\zeta}\left(\frac{F(\eta)}{F(0)}\right)\right]_{\zeta=0} .$$

In the same way,

$$\omega^* = \sqrt{F^*(0)}\left[\frac{d}{d\zeta}\left(\frac{F^*(\eta)}{F^*(0)}\right)\right]_{\zeta=0} .$$

Since $\left[\frac{d}{d\zeta}\,\frac{F(\eta)}{F(0)}\right]_{\zeta=0} = \left[\frac{d}{d\zeta}\left(\frac{F^*(\eta)}{F^*(0)}\right)\right]_{\zeta=0}$,

it follows that

$$\frac{\omega^*}{\omega} = \sqrt{\frac{F^*(0)}{F(0)}} = \sqrt{\frac{\theta_1^* + \lambda}{\theta_1 + \lambda}} \quad . \qquad [7]$$

In the report, the case $\beta = 0$ or $\lambda = 0$ was treated. Hence the conversion formula for this case is

$$\omega^* = \omega\sqrt{\theta_1^* \, \theta_1} \quad . \qquad [8]$$

In particular, if $\theta_1^* = 1$,

$$\omega = \omega^*\sqrt{\theta_1} \quad . \qquad [9]$$

(D) Outflow Case for the Linear Diffusivity Function.

Here, $D_2 = \alpha\theta_2 + \beta$, $\eta = \zeta\sqrt{D_2/\alpha}$, where $\zeta = x/\sqrt{2D_2 t}$; and $F(0) = \theta_2 + \lambda$. As before

$$\omega^* = \omega\sqrt{\frac{F^*(0)}{F(0)}} = \omega\sqrt{\frac{\theta_2^* + \lambda}{\theta_2 + \lambda}} \quad . \qquad [10]$$

For the case treated here, $\lambda = 0$, so that

$$\omega^* = \omega\sqrt{\theta_2^* / \theta_2} \quad . \qquad [11]$$

In the particular case $\theta_2^* = 1$,

$$\omega = \omega^*\sqrt{\theta_2} \quad . \qquad [12]$$

Construction of Graphs in Figs. 8 and 11

(A) Inflow Case for the Exponential Function.

Using the original data, corresponding to a given boundary water content $\theta = \theta_1^{**} \neq 0$ and a given slope $\omega = \omega^{**}$, an initial water content $\theta = \theta_o^{**}$ was determined. The values $\theta = \theta_1^{**}$ and $\theta = \theta_o^{**}$ thus determined the ratio

$$\frac{D_1}{D_o} = e^{\beta(\theta_1^{**} - \theta_o^{**})} = \frac{f(0)}{f(\infty)} \quad . \qquad [1]$$

By varying the slope $\omega = \omega^{**}$, various ratios D_1/D_o were computed. However, in view of the fact that the ratio D_1/D_o is the same whenever the difference $\theta_1^{**} - \theta_o^{**}$ = constant (e.g., for $\theta_1^{**} = 0.20$, $\theta_o^{**} = 0.10$ as well as for $\theta_1^{**} = 0.10$, $\theta_o^{**} = 0.00$), it was decided to use zero boundary water content in this case as a standard reference point. To determine the corresponding change in the slope ω, appeal was made to formula 3 of Appendix I.

If ω^{**} corresponds to θ_o^{**} and ω^* corresponds to θ_o^*, this formula reads

$$\omega^* = \omega^{**} \sqrt{e^{\beta(\theta_o^* - \theta_o^{**})}} \quad . \qquad [2]$$

Therefore, for $\theta_o^* = 0$, formula 2 becomes

$$\omega^* = \omega^{**} \Big/ \sqrt{e^{\beta \theta_o^{**}}} \quad . \qquad [3]$$

It was this formula which was used to draw the graph in Fig. 8 of ω^* versus D_1/D_o.

Suppose a slope ω is desired corresponding to any given values $\theta = \theta_1$ and $\theta = \theta_o$. Then, these values determine the ratio D_1/D_o which in turn determines ω^* corresponding to $\theta_o^* = 0$ from Fig. 8. The new adjusted slope now follows from formula 4, Appendix I, namely,

$$\omega = \omega^* \sqrt{e^{\beta \theta_o}} \quad . \qquad [4]$$

(B) Outflow Case for the Exponential Function.

Using the original data, corresponding to a given boundary water content $\theta = \theta_2^{**} \neq 0$ and a given slope $\omega = \omega^{**}$, an initial water content $\theta = \theta_o^{**}$ was determined. The values $\theta = \theta_2^{**}$ and $\theta = \theta_o^{**}$ thus determined the ratio

$$\frac{D_o}{D_2} = e^{\beta(\theta_o^{**} - \theta_2^{**})} = \frac{f(\infty)}{f(0)} \quad . \qquad [1]$$

By varying the slope $\omega = \omega^{**}$, various ratios D_o/D_2 were computed. However, in view of the fact that the ratio D_o/D_2 is the same whenever the $\theta_o^{**} - \theta_2^{**}$ = constant, (e.g., for $\theta_o = 0.20$, $\theta_1 = 0.10$ as well as for $\theta_o = 0.10$, $\theta_2 = 0.00$), it was decided to use zero boundary water content in this case as a standard reference point. To determine the corresponding change in the slope ω, appeal was made to formula 5 of Appendix I. If ω^{**} corresponds to θ_2^{**} and ω^* corresponds to θ_2^*, this formula reads

$$\omega^* = \omega^{**} \sqrt{e^{\beta(\theta_2^* - \theta_2^{**})}} \quad . \qquad [2]$$

Therefore, for $\theta_2^* = 0$, formula 2 becomes

$$\omega^* = \omega^{**} \Big/ \sqrt{e^{\beta \theta_2^{**}}} \quad . \qquad [3]$$

It was this formula which was used to draw the graph in Fig. 8 of ω^* versus D_o/D_2.

Suppose a slope ω is desired corresponding to any given values $\theta = \theta_2$ and $\theta = \theta_o$. Then, these values determine the ratio D_o/D_2 which in turn determines ω^* corresponding to $\theta_2^* = 0$ from Fig. 8. The new adjusted slope now follows from formula 6, Appendix I, namely,

$$\omega = \omega^* \sqrt{e^{\beta \theta_2}} \quad . \qquad [4]$$

(C) Inflow Case for the Linear Diffusivity Function $D = \alpha\theta$ ($\beta = 0$).

From the original data, corresponding to a given boundary water content $\theta = \theta_1^{**} \neq 0$ and a given slope $\omega = \omega^{**}$, an initial water content $\theta = \theta_o^{**}$ was determined. The values $\theta = \theta_1^{**}$ and $\theta = \theta_o^{**}$ thus determined the ratio

$$\frac{D_1}{D_o} = \frac{\theta_1^{**}}{\theta_o^{**}} = \frac{F(0)}{F(\infty)} \quad . \qquad [1]$$

By varying the slope $\omega = \omega^{**}$, various ratios D_1/D_o were computed. Here, in view of equation 1, a given ratio D_1/D_o = constant will correspond to all problems in which the ratio $\theta_1^{**}/\theta_o^{**}$ is equal to this same constant. In this case, since a zero water constant as a base could not be used, it was decided to base ω on a water content of unity.

From formula 8, Appendix I,

$$\omega^* = \omega^{**} \sqrt{\theta_1^*/\theta_1^{**}} \quad . \qquad [2]$$

For $\theta_1^* = 1$, this formula becomes

$$\omega^* = \omega^{**} \Big/ \sqrt{\theta_1^{**}} \quad . \qquad [3]$$

The graph in Fig. 11 was constructed on the basis of this formula.

Suppose a slope ω is desired corresponding to any given values $\theta = \theta_1$ and $\theta = \theta_o$. Then, these values determine the ratio D_1/D_o which in turn determines ω^* corresponding to $\theta_1^* = 1$ from fig. 11. The new adjusted slope ω now follows from formula 9, Appendix I, namely,

$$\omega = \omega^* \sqrt{\theta_1} \quad .$$

(D) Outflow Case for the Linear Diffusivity Function $D = \alpha\theta$ ($\beta = 0$).

In this case the ratio

$$\frac{D_o}{D_2} = \frac{\theta_o^{**}}{\theta_2^{**}} = \frac{F(\infty)}{F(0)}$$

was computed. As for inflow, unity was taken as the base for computing ω. From formula 11, Appendix I,

$$\omega^* = \omega^{**} \sqrt{\theta_2^*/\theta_2^{**}} \quad ,$$

which for $\theta_2^* = 1$ becomes

$$\omega^* = \omega^{**}/\sqrt{\theta_2^{**}} \quad .$$

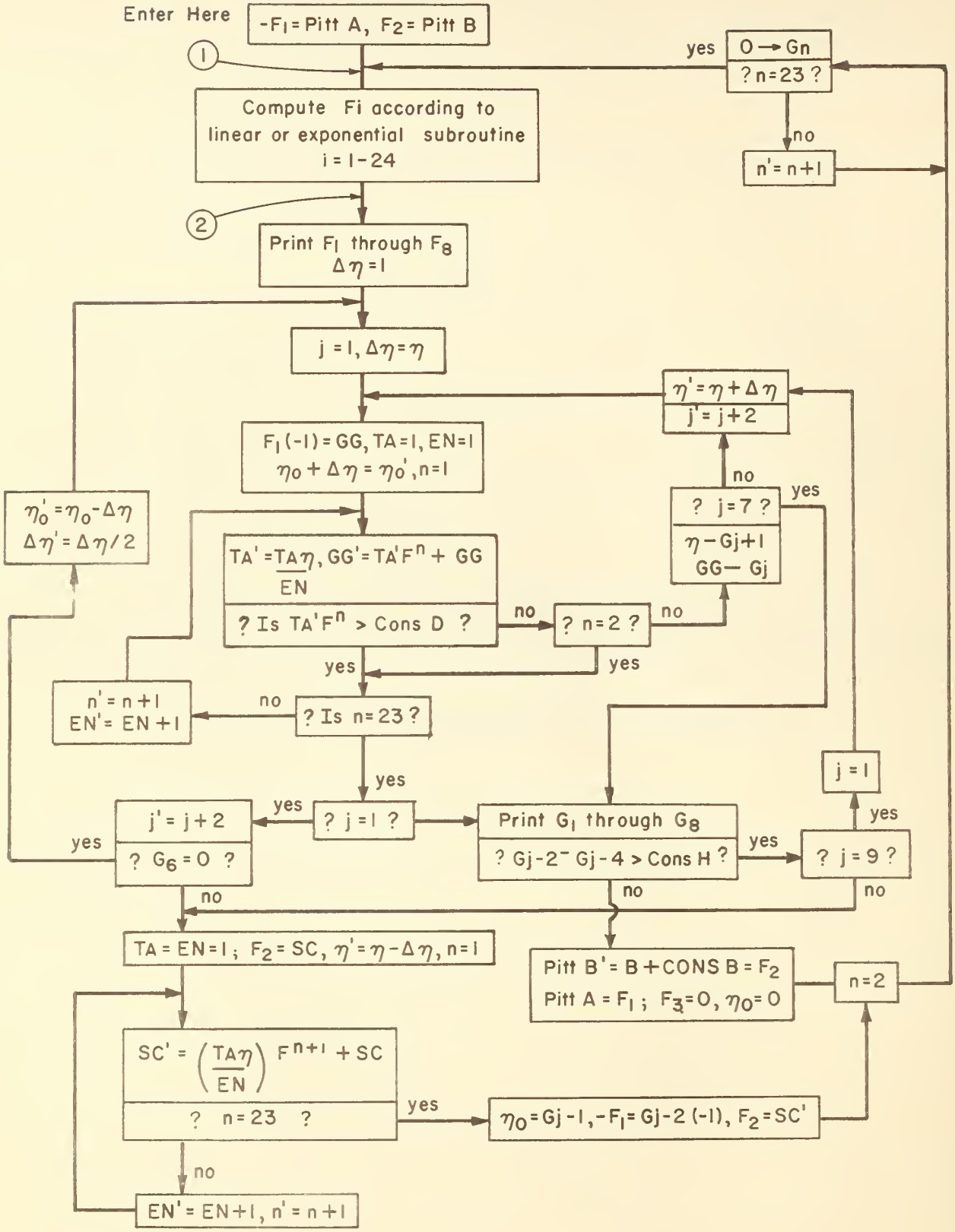
Formula 3 was used to construct Fig. 11.

The slope ω corresponding to any given values $\theta = \theta_2$, $\theta = \theta_o$ is obtained as follows. Using the preceding values, determine the ratio D_o/D_2. From fig. 11 obtain the value of ω^* corresponding to $\theta_2^* = 1$. The new adjusted slope ω now follows from formula 12, Appendix I,

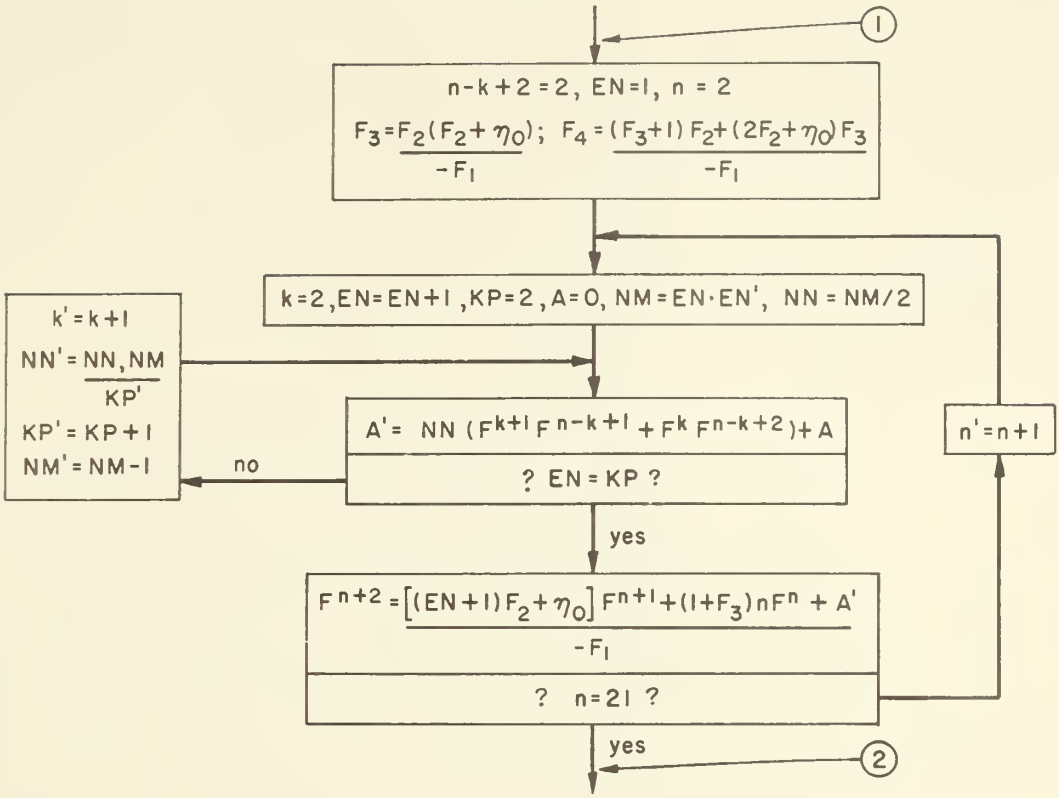
$$\omega = \omega^* \sqrt{\theta_2} \quad .$$

APPENDIX III

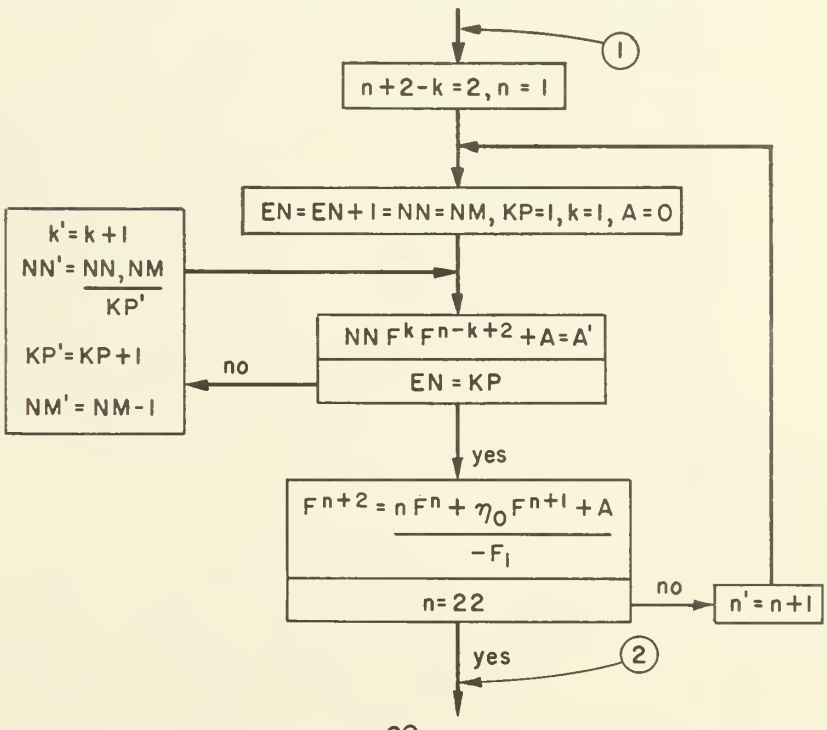
Flow Diagrams Outlining the Computational Program


Enter Here
-F1 = Pitt A, F2 = Pitt B
1
yes
0 → Gn
? n=23 ?
no
n' = n+1
Compute Fi according to linear or exponential subroutine i = 1-24
2
Print F1 through F8
Δη = 1
j = 1, Δη = η
η' = η + Δη
j' = j+2
F1(-1) = GG, TA = 1, EN = 1
η0 + Δη = η0', n = 1
no
yes
? j = 7 ?
η - Gj+1
GG - Gj
η0' = η0 - Δη
Δη' = Δη/2
TA' = TAη/EN, GG' = TA'F^n + GG
? Is TA'F^n > Cons D ?
no
? n=2 ?
no
yes
yes
n' = n+1
EN' = EN+1
no
? Is n=23 ?
yes
j = 1
yes
j' = j+2
yes
? j=1 ?
Print G1 through G8
? Gj-2 - Gj-4 > Cons H ?
yes
? j = 9 ?
yes
yes
? G6 = 0 ?
no
no
no
TA = EN = 1; F2 = SC, η' = η - Δη, n = 1
Pitt B' = B + CONS B = F2
Pitt A = F1; F3 = 0, η0 = 0
n=2
SC' = (TAη/EN) F^n+1 + SC
? n = 23 ?
yes
η0 = Gj-1, -F1 = Gj-2 (-1), F2 = SC'
no
EN' = EN+1, n' = n+1

LINEAR DIFFUSIVITY SUBROUTINE



EXPONENTIAL DIFFUSIVITY SUBROUTINE



Growth Through Agricultural Progress